Indian Institute of Science

Pravega 2022

Enumeration - Prelims

July 16, 2022

Timing: 2:00 PM to 6:00 PM | Max mark: 90

Objectives

1. In the land of *Lagneb*, there are 2022 villages $V_1, V_2, \ldots V_{2022}$, located such that $V_1V_2 \ldots V_{2022}$ is a regular polygon. *Sakjit* the dacoit is a resident of village V_1. He raids the village V_1 $(= U_0)$, and from there he travels to a village U_1 $(= V_i$ for some $i \neq 1)$ and raids it. From village U_i, he then travels to another village U_{i+1} and raids it. However, because he gets tired in the process, the distance he travels between consecutive raids is reduced $(U_iU_{i-1} > U_{i+1}U_i)$. Find the maximum number of villages he can raid.
Note: *Sakjit* may raid the same village twice, at different instances. **(3)**

2. Let ABC be an acute triangle with symmedian point K and circumcenter O. Suppose that OB is tangent to the circumcircle of $\triangle BKC$. If $BC = 13, CA = 19$, find AB. **(3)**

3. Let $p(x) = \sum_{i=0}^{2022} a_i x^i$ be a real polynomial, and suppose that $a_{2021} = a_{2020} = 0$. Suppose that $a_{2019}, a_{2018}, \ldots, a_0$ are not all zero. Find the maximum number of real roots of $p(x)$, counting multiplicity. **(3)**

4. Lily pads numbered from 1 to 2022 are placed on a pond. *Bruno* the frog starts on pad 1. Every moment he picks a integer greater than the integer on the pad he is standing on, uniformly at random, and jumps to the corresponding lily pad. The probability he lands on the pad numbered 2000 at some instant is $\frac{a}{b}$, where a, b are coprime positive integers. Find the value of $a \times b$. **(3)**

5. Let $f(x) = (x^2 - 5)(x^2 - 7)(x^2 - 35)$. For $n \in \mathbb{N}$, let $A(n)$ denote the number of $0 < m \leq n$ such that $f(x) \equiv 0(\bmod m)$ has no integral solution. Let $r = \lim_{n\to\infty} \frac{A(n)}{n}$. Compute $\lfloor \frac{1}{r} \rfloor$. **(3)**

6. Let $A_1A_2 \ldots A_{2n}$ be a regular polygon with $2n$ sides. ($n \geq 1011$ is a positive integer). Suppose that
$$2 \cdot A_1A_2 \cdot A_1A_{2022} = A_1A_3 \cdot A_1A_{n+1}$$
Find the sum of all possible values of n.

7. Find the length of the smallest repeating part in the decimal expansion of $\frac{1}{83 \times 107}$ **(3)**

8. Let $ABCD$ be a square in the Cartesian plane with coordinates $A = (0,0)$, $B = (10,0)$, $C = (10,10)$ and $D = (0,10)$. Find the number of parallelograms $PQRS$ inscribed in square $ABCD$, such that P, Q, R, and S have integer coordinates and $\{P, Q, R, S\} \cap \{A, B, C, D\} = \phi$. **(3)**

9. Let $ABCD$ be a cyclic quadrilateral with $AD \neq BC$, and E is the intersection of diagonals AC and BD. Suppose that the perpendiculars from E to AB and CD bisect segments CD and AB respectively. Given $AB = 17$, $BC = 13$, and $CD = 29$, find the area of $ABCD$. **(3)**

10. Primes a, b, c, d satisfy the following equation.

$$\left(\cos \frac{2\pi}{7}\right)^{\frac{1}{3}} + \left(\cos \frac{4\pi}{7}\right)^{\frac{1}{3}} + \left(\cos \frac{6\pi}{7}\right)^{\frac{1}{3}} = \left(\frac{a - b \times \sqrt[3]{c}}{d}\right)^{\frac{1}{3}}$$

Find the value of $a + b + c + d$. **(3)**

Subjectives

1. *Bhavya* wants to do a geometry problem, so he draws on a blackboard a triangle ABC. He draws the altitude AD from A to BC. He then marks the incenters I_B and I_C of $\triangle ADB$ and $\triangle ADC$, and goes for a break. While *Bhavya* is away, the troll *Sudharshan* erases the blackboard except for points A, I_B and I_C. Can you help *Bhavya* restore the triangle ABC (using only straightedge and compass constructions)? **(10)**

2. Show that the equation $x^3+y^3=z^2$ has infinitely many solutions in positive integers with $\gcd(x,y,z)=1$. **(10)**

3. Let $u : \mathbb{Z}^5 \to \mathbb{Z}$, be an integer valued function. Suppose that

$$u(a,b,c,d,e) - u(b,a,c,d,e) + u(b,c,a,d,e) - u(b,c,d,a,e) + u(b,c,d,e,a) = 0$$

and

$$u(a,b,c,d,e) - u(a,c,b,d,e) + u(a,c,d,b,e) - u(a,c,d,e,b) + u(c,a,b,d,e)$$
$$-u(c,a,d,b,e) + u(c,a,d,e,b) + u(c,d,a,b,e) - u(c,d,a,e,b) + u(c,d,e,a,b) = 0$$

Prove that $u(a,b,c,d,e) = u(e,d,c,b,a)$. **(10)**

4. A (i,j) shuffle permutation σ of $\{1,2,\dots,i+j\}$ is a permutation for which $\sigma(1) < \sigma(2) < \cdots < \sigma(i)$ and $\sigma(i+1) < \sigma(i+2) \cdots < \sigma(i+j)$. Let $d(i,j)$ be the difference between the number of even (i,j) shuffle permutations, and the number of odd (i,j) shuffle permutations. Prove that, $d(a,b)^2 > \left(1+\frac{(b-1)}{a}\right)^{(a-1)}$ if and only if ab is even. **(10)**

5. Let $ABCD$ be a quadrilateral. Do there exist points P, Q, R, S such that A, B, C and D are the respective circumcenters of $\triangle QRS$, $\triangle RSP$, $\triangle SPQ$ and $\triangle PQR$? **(10)**

6. Let k be a positive integer, and let ϵ be a positive real number. Prove that there are infinitely many positive integers n, such that the largest prime factor of n^k+1 is less than n^ϵ. **(10)**

Best wishes