Enumeration 2023 Prelims
Solutions Document

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Contents

1 Problems

1.1 Computational Problems

1. Let x and y be positive real numbers satisfying $x + y = 1$. The maximum possible value of $x^4y + xy^4$ can be expressed as $\frac{a}{b}$ for relatively prime positive integers a, b. Compute $a + b$.

2. Anna writes a sequence of positive integers $(a_1, a_2, \cdots, a_n)$, such that

$$a_1 + a_2 + \cdots + a_n = 2023$$

   Suppose the sequence Anna picks is equally likely to be any sequence satisfying the above condition. Find the expected value of n.

   Note: The *expected value* of n is the average value of n across all possibilities.

3. For a positive integer n relatively prime to 2023, we define its *order* modulo 2023, to be the smallest positive integer d such that 2023 divides $n^d - 1$. Find the number of integers n relatively prime to 2023 such that $1 \leq n \leq 2023$ and the *order* of n modulo 2023 is 136.

4. Let $S = \displaystyle\sum_{i=0}^{50}\sum_{j=0}^{50}(-1)^{i+j}\binom{100}{i+j}$. Given that S is a positive integer, find the highest exponent of 2 dividing S.

5. Let $ABCD$ be a *bicentric* quadrilateral, that is, it has both an incircle and a circumcircle. Suppose the incircle of $ABCD$ is tangent to BC at X. If $AD = 5$, $BX = 4$ and $CX = 6$, then the area of $ABCD$ can be expressed as $a\sqrt{b}$ where a, b are positive integers and b isn't divisible by the square of any prime. Compute ab.

6. An ice cream parlour with infinitely many identical stalls allows one customer to enter every minute. Upon entry, a customer goes to an empty stall and stays there till their order is delivered, and then leaves the parlor instantly. The probability that it takes exactly n minutes to prepare their order is $\frac{1}{n(n+1)}$. Let T denote the expected amount of time after the first customer walks in, for *someone's* order to be delivered. Find $\lfloor 10T \rfloor$.

   Note : $\lfloor x \rfloor$ is the smallest integer not exceeding x.

7. Let O denote the circumcenter of ΔABC with $AB = \sqrt{3}$ and $AC = 2$. Suppose the circumcircle of $\triangle BOC$ intersects AB and AC again at X, Y, respectively. If XY is tangent to the circumcircle of triangle ΔABC, then BC^2 can be expressed as $\frac{a+b\sqrt{c}}{d}$, where a, b, c, d are positive integers such that $\gcd(a, b, d) = 1$ and c isn't divisible by the square of any prime. Compute $a + b + c + d$.

8. Find the number of ways of placing 4 knights on a 4×4 board such that for every knight, there is an unique knight attacking it.

9. Let P_1, P_2, P_3 be three monic quadratic polynomials, which are pairwise unequal and have no double

root. Suppose for every $i = 1, 2, 3$, there exists an unique real number $a_i \neq 1$ such that the equation

$$P_i(x) = a_i P_{i+1}(x)$$

has exactly one solution x_i (where $P_4 \equiv P_1$). If $x_1 = 3, a_1 = \frac{-2}{3}$ and $a_2 = \frac{3}{5}$, find the sum of all possible values of $x_2 + x_3$.

10. Circles ω_1 and ω_2 with centres O_1 and O_2, and radii $13, 15$ respectively, intersect at points X, Y. Points P, Q lie on ω_1, ω_2 respectively such that P, Y, Q are collinear in that order. Suppose PO_1 and QO_2 intersect at T. If $XT \parallel PQ$, and $O_1O_2 = 14$, then the circumradius of $\triangle XPQ$ can be expressed as $\frac{a}{b}$, for relatively prime positive integers a, b. Compute $a + b$.

11. Consider the polynomial :

$$P(x) = \prod_{1 \leq a \leq 101} (x - a^5)$$

Let $Q(x)$ be the remainder obtained when $P(x)$ is divided by $x^3 - 1$. Find the remainder when $Q(2)$ is divided by 101.

12. Let A, B be points on an eliipse $\mathcal{E}$ with focii F_1, F_2. Suppose AF_2 intersects BF_1 at C and AF_1 intersects BF_2 at D. The tangents to $\mathcal{E}$ at A, B intersect at P. Suppose C, D, F_1, F_2 lie on a circle. If $PF_1 = 5$, $F_1F_2 = 7$ and $PF_2 = 8$, then CD can be expressed as $\frac{a}{b}$ for relatively prime positive integers a, b. Compute $a + b$.

13. Let a, b, c, d be real numbers such that $abcd = -1$ and the following equations hold :

$$|d(a-b)(b-c)(c-a)| = 1$$

$$|a(b-c)(c-d)(d-b)| = 2$$

$$|b(c-a)(a-d)(d-c)| = 6$$

Find the sum of all possible values of $|c(d-a)(a-b)(b-d)|$.

14. A positive integer n is said to be a *quadratic residue modulo 97*, if there exists an integer m, such that $m^2 - n$ is divisible by 97 and $97 \nmid n$. Alice randomly picks a triple (a, b, c) of *quadratic residues modulo 97*, where $1 \leq a, b, c \leq 97$ and her choice of triple is equally likely among all possible choices. If the probability that $a + b + c$ is **NOT** a *quadratic residue modulo 97* can be expressed as $\frac{x}{y}$, for relatively prime positive integers x, y, then compute the value of $x + y$.

15. Let N denote the number of upright paths from $(0, 0)$ to $(10, 10)$ which intersect the line $x = y$ at exactly 5 points other than the start and end points. If N can be expressed as $2^a \times b$, where b is an odd positive integer, compute $a + b$.

    Note : A upright path is one where we only take steps towards right or upwards. For example the following is an upright path from $(0, 0)$ to $(2, 2)$:

$$(0, 0) \to (1, 0) \to (2, 0) \to (2, 1) \to (2, 2)$$

whereas the following is not (as it has a leftward step):

$$(0,0) \to (1,0) \to (2,0) \to (3,0) \to (3,1) \to (3,2) \to (2,2)$$

1.2 Subjective Problems

1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that the following equation holds for all $x, y \in \mathbb{R}$:

$$f(yf(x)) + f(xy) = 2yf(x)$$

2. Find all pairs of positive integers (a, b) such that for all positive integers $n > 2023^{2023}$, the number $n^2 + an + b$ has a divisor $d > 1$, such that $n \mid d - 1$.

3. Let ω denote the circumcircle of triangle ΔABC. Suppose the internal and external bisectors of $\angle BAC$, intersect $\overline{BC}$ and ω again at K, L respectively. Points X, Y lie on ω such that $LK = LX = LY$. Prove that $\overline{XY}$, and the line through K perpendicular to $\overline{BC}$ meet on the A-median.

4. 2002 people stand in a line. Each person either always tells the truth, or always lies. Starting from the back, Alice asks 1995 people :

   How many liars are are standing in front of you?

   and records their answers. Prove Alice can pick a subset of non-negative integers $\mathcal{S}$ such that the sum of elements of $\mathcal{S}$ is atmost 2023 and she can guarantee that number of truthful people is in $\mathcal{S}$.

5. The A-excircle of ΔABC is tangent to $\overline{BC}$ at D. The line $\overline{AD}$ intersects the incircle of ΔABC at points Y, Z such that $AZ > AY$. The line through Z parallel to the external angle bisector of $\angle BAC$ meets $\overline{BC}$ at X. Prove that $\overline{XY}$ passes through the midpoint of arc $\widehat{BAC}$ in the circumcircle of ΔABC.

2 Answer Key to Computational Problems

1. 13
2. 1012
3. 128
4. 4
5. 90
6. 17
7. 25
8. 128
9. 7
10. 67
11. 83
12. 1209
13. 12
14. 3455
15. 435

3 Solutions

3.1 Computational Problems

These are intended to be solution sketches, rather than fully written out proofs.

Problem 1

Let x and y be positive real numbers satisfying $x + y = 1$. The maximum possible value of $x^4y + xy^4$ can be expressed as $\frac{a}{b}$ for relatively prime positive integers a, b. Compute $a + b$.

Sol : Write $xy(x^3 + y^3) = xy(1 - 3xy) \leq \frac{1}{12}$. This leads to an answer of $\boxed{13}$.

Problem 2

Anna writes a sequence of positive integers $(a_1, a_2, \cdots, a_n)$, such that

$$a_1 + a_2 + \cdots + a_n = 2023$$

Suppose the sequence Anna picks is equally likely to be any sequence satisfying the above condition. Find the expected value of n.

Note: The *expected value* of n is the average value of n across all possibilities.

Sol : We may think of this in terms of putting a positive integer amount of "balls" in each of the a_i, so that we have 2023 balls in the end. We want to find the expected value of partitions. Lay out all the balls in the straight line, and note that there are 2022 "spaces" between each two. As the partitions are picked randomly, every space has a $\frac{1}{2}$ chance of getting picked, leading to expected value of spaces picked being 1011. The number of partitions is one more than that and equal to $\boxed{1012}$.

Problem 3

For a positive integer n relatively prime to 2023, we define its *order* modulo 2023, to be the smallest positive integer d such that 2023 divides $n^d - 1$. Find the number of integers n relatively prime to 2023 such that $1 \leq n \leq 2023$ and the *order* of n modulo 2023 is 136.

Sol : The multiplicative group $(\mathbb{Z}/2023\mathbb{Z})^\times$ is just $(\mathbb{Z}/7\mathbb{Z})^\times$ (call this A, with primitive root a) times $(\mathbb{Z}/289\mathbb{Z})^\times$ (call this B, with primitive root b). Note that $\phi(2023) = 6 \times 16 \times 17$, and we want order =136. Now each element is just of the form $a^x b^y$, and orders are just a function of x, y. So we want to multiply factors of 6 and 16×17 to get 17×8? The only ways are:

order 1, 17*8 from A,B resp, AND order 2, 17*8 from A,B resp

So we want $a^x b^y$ such that $\text{ord}_7(a^x) = 1$ or 2 i.e. $a^2 = 1$ and $\text{ord}_{289}(b^y) = 17 \times 8 = (17 \times 16)/2$, so the square of any primitive root works. That's $\phi(17 \times 8) = 16 \times 4 = 64$, so a total of $2 \times 64 = \boxed{128}$ solutions.

Problem 4

Let $S = \displaystyle\sum_{i=0}^{50}\sum_{j=0}^{50}(-1)^{i+j}\binom{100}{i+j}$. Given that S is a positive integer, find the highest exponent of 2 dividing S.

Sol : Write as $\displaystyle\sum_{n=0}^{50}\{\min(n, 100-n)+1\}\cdot(-1)^n\binom{100}{n}$. So this becomes equivalent to finding the coefficient of x^{100} in

$$(1-x)^{100}(1+2x+3x^2+\cdots+50x^{49}+51x^{50}+50x^{51}+\cdots 2x^{99}+x^{100})$$

The latter expression is $(1+x+x^2+\cdots+x^{50})^2$, and hence $S = 2\times\binom{98}{49}$. One can now calculate $v_2(S) = \boxed{4}$.

Problem 5

Let $ABCD$ be a *bicentric* quadrilateral, that is, it has both an incircle and a circumcircle. Suppose the incircle of $ABCD$ is tangent to BC at X. If $AD = 5$, $BX = 4$ and $CX = 6$, then the area of $ABCD$ can be expressed as $a\sqrt{b}$ where a, b are positive integers and b isn't divisible by the square of any prime. Compute ab.

Sol : Let t_a denote the length of tangent from A to the incircle. Define t_b, t_c, t_d analogously for the other vertices. Then we claim that $t_at_c = t_bt_d$. Suppose the incircle is tangent to AB at P and CD at Q. Then note that $\triangle PAI \sim \triangle QIC$ and $PBI \sim QID$. So we have : $\frac{AP}{BP} = \frac{AP}{PI}\cdot\frac{PI}{PB} = \frac{QI}{QC}\cdot\frac{QD}{QI} = \frac{QD}{QC}$ which implies the claim. Now one can find that the sidelengths are $10, 5, 6, 9$, which gives an area of $30\sqrt{3}$. Thus the answer is $\boxed{90}$.

Problem 6

An ice cream parlour with infinitely many identical stalls allows one customer to enter every minute. Upon entry, a customer goes to an empty stall and stays there till their order is delivered, and then leaves the parlor instantly. The probability that it takes exactly n minutes to prepare their order is $\frac{1}{n(n+1)}$. Let T denote the expected amount of time after the first customer walks in, for *someone's* order to be delivered. Find $\lfloor 10T\rfloor$.

Note : $\lfloor x\rfloor$ is the smallest integer not exceeding x.

Sol : Note that the probability that a customer is in the store after k minutes is $1-\sum_{i=0}^{k}\frac{1}{i(i+1)} = \frac{1}{k+1}$. Now note, that at the n-th minute, there are $n+1$ customers, and the j-th customer is in the store for $n+1-j$ minutes. Thus $\Pr(T>n) = \prod_{j=0}^{n+1}\frac{1}{n+1-j} = \frac{1}{(n+1)!}$ Hence we have :

$$\mathbb{E}[T] = \sum_{n\geq 0}\Pr(T>n) = \sum_{n\geq 0}\frac{1}{(n+1)!} = e-1$$

So the answer is $\lfloor e-1\rfloor = \boxed{17}$

Problem 7

Let O denote the circumcenter of ΔABC with $AB = \sqrt{3}$ and $AC = 2$. Suppose the circumcircle of $\triangle BOC$ intersects AB and AC again at X, Y, respectively. If XY is tangent to the circumcircle of triangle ΔABC, then BC^2 can be expressed as $\frac{a+b\sqrt{c}}{d}$, where a, b, c, d are positive integers such that $\gcd(a, b, d) = 1$ and c isn't divisible by the square of any prime. Compute $a + b + c + d$.

Sol : Note that XY is antiparallel to BC (as $XYBC$ is cyclic), and hence $AO \perp XY$. Hence the tangency is at the A-antipode of $\triangle ABC$. Also one may prove via angle chasing, that O is the orthocenter of $\triangle AXY$. Thus we have $ABCH \sim AYXO$, and also we know that H lies on the A-midline in $\triangle ABC$. This gives the equation ($S_A = \frac{b^2+c^2-a^2}{2}$, etc) :

$$S_B S_C = S_A(S_B + S_C) \implies \frac{1}{4}(a^2-1)(a^2+1) = \frac{1}{2}a^2(7-a^2)$$

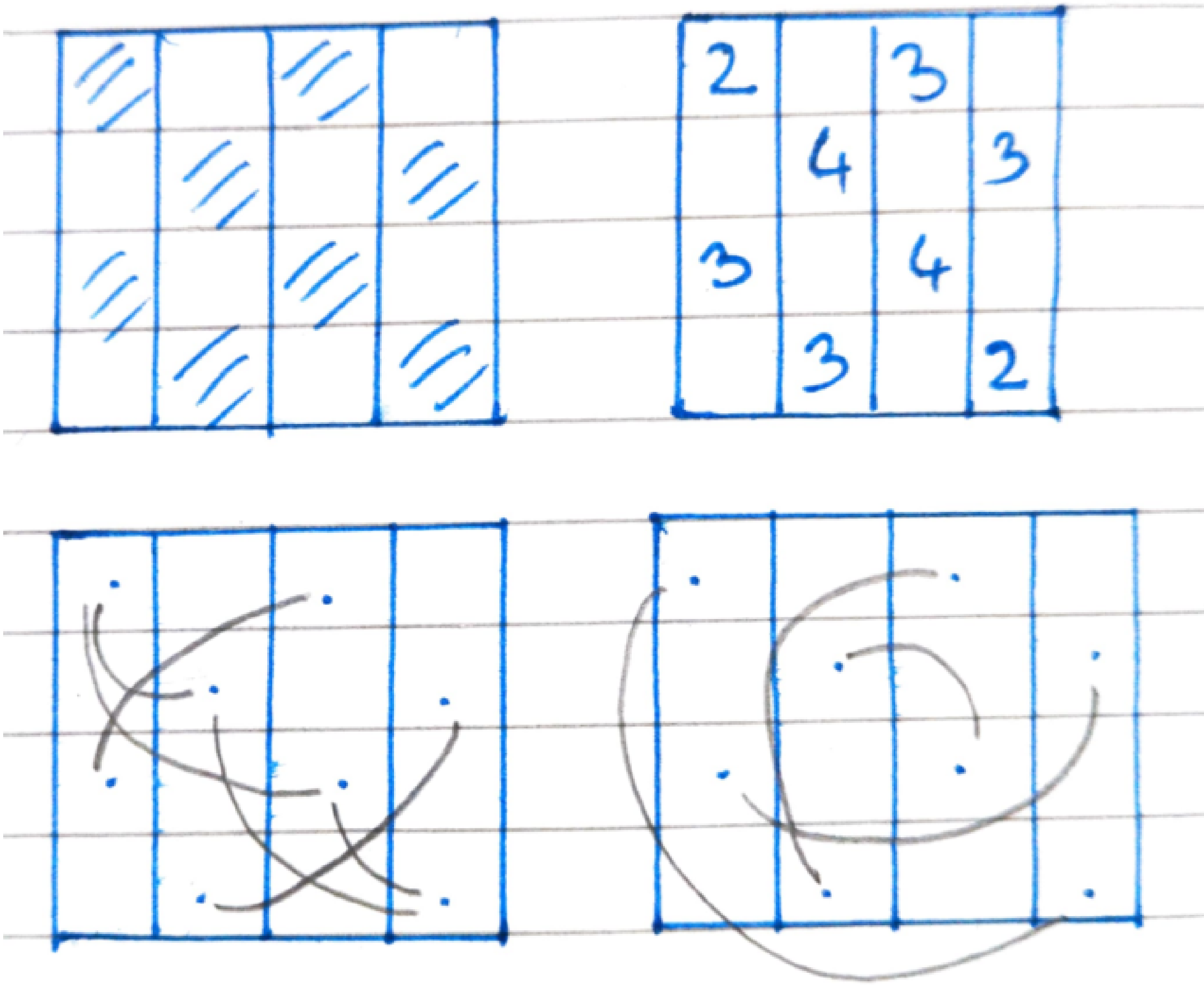
$$\implies 3a^4 - 14a^2 - 1 = 0 \implies a = \sqrt{\frac{1}{3}(7+2\sqrt{13})}$$

This gives a value of $\boxed{25}$.

Problem 8

Find the number of ways of placing 4 knights on a 4×4 board such that for every knight, there is an unique knight attacking it.

Sol :(by **Agamjeet Singh and Malay Mahajan**)



We colour the board as shown below. No knight in the shaded area can attack any other knight in the shaded area. Same goes for the unshaded area. Also each knight is, in some sense, paired with another knight (They are attacking each other). So there are 2 knights in shaded area and 2 knights in unshaded area.

Now we consider all pairs of knights on the shaded area and consider the number of ways in which we can place the other 2 knights in the unshaded area. From what follows, 'sharing a knight' means if both the knights in the shaded area attack a particular knight in the unshaded area. In the diagram, I have numbered each shaded cell with the number of knights in the unshaded area that it can attack.

Two pairs would either share nothing, share one knight or share two knights. The first diagram shows sharing no knights and the other one shows sharing two knights. In all other cases, exactly one knight is shared.

We count from the top row, going down, from left to right (First fixing one cell, then varying the other cell)

$1 \cdot 2 + 2 \cdot 4 + 1 \cdot 2 + 1 \cdot 2 + 2 \cdot 4 + 1 \cdot 2 + 0 + 2 \cdot 3 + 2 \cdot 2 + 3 \cdot 3 + 2 \cdot 3 + 1 \cdot 1 + 2 \cdot 1 + 3 \cdot 2 + 3 \cdot 2 + 2 \cdot$

$2+3\cdot 2+4\cdot 2+1\cdot 1+2\cdot 3+3\cdot 3+2\cdot 1+2\cdot 3+2\cdot 2+2\cdot 1+3\cdot 2+4\cdot 2+2\cdot 1=\boxed{128}$

Problem 9

Let P_1, P_2, P_3 be three monic quadratic polynomials, which are pairwise unequal and have no double root. Suppose for every $i = 1, 2, 3$, there exists an unique real number $a_i \neq 1$ such that the equation

$$P_i(x) = a_i P_{i+1}(x)$$

has exactly one solution x_i (where $P_4 \equiv P_1$). If $x_1 = 3$, $a_1 = \frac{-2}{3}$ and $a_2 = \frac{3}{5}$, find the sum of all possible values of $x_2 + x_3$.

Sol : One can prove that for two quadratics P, Q which arent constant multiples of each other, and do not have a double root, then there exists a $k \in \mathbb{R}$, such that $P - kQ$ has exactly one real root, iff P, Q share exactly one root in common. This is just a discriminant calculation. This implies that all three quadratics in question share one pair of roots each. With this in mind, its easy to see $x_2 = 6$ and $x_3 = 1$, leading to an answer of $\boxed{7}$.

Problem 10

Circles ω_1 and ω_2 with centres O_1 and O_2, and radii $13, 15$ respectively, intersect at points X, Y. Points P, Q lie on ω_1, ω_2 respectively such that P, Y, Q are collinear in that order. Suppose PO_1 and QO_2 intersect at T. If $XT \parallel PQ$, and $O_1O_2 = 14$, then the circumradius of $\triangle XPQ$ can be expressed as $\frac{a}{b}$, for relatively prime positive integers a, b. Compute $a + b$.

Sol : First note by Salmon's lemma that $XO_1O_2 \sim XPQ$, so that $T \in \odot(XPQ)$. Then let O denote the circumcenter of $\triangle XPQ$. One can prove the following using angle chasing :

- X, O, Y are collinear.
- $XO \perp O_1O_2$
- X, O, O_1, O_2 lie on a circle

Using this and power of point, we may conclude that $XO = \frac{63}{4}$, leading to an answer of $\boxed{67}$.

Problem 11

Consider the polynomial :

$$P(x) = \prod_{1 \leq a \leq 101} (x - a^5)$$

Let $Q(x)$ be the remainder obtained when $P(x)$ is divided by $x^3 - 1$. Find the remainder when $Q(2)$ is divided by 101.

Sol : Work in $\mathbb{F}_{101}[X]$. We have :

$$P(x) = x \prod_{a \in A} (x - a)^5$$

where A denotes the set of all non zero fifth powers modulo 101. Note that $\prod_{a \in A}(x-a) = (x^{\frac{100}{5}} - 1) = (x^{20}-1)$. Thus we need to find $x(x^{20}-1)^5 \equiv x(x^2-1)^5$ modulo x^3-1. This is found to be $9x - 9x^2$. Plugging in $x = 2$ and taking the positive remainder, gives $\boxed{83}$.

> **Problem 12**
>
> Let A, B be points on an eliipse $\mathcal{E}$ with focii F_1, F_2. Suppose AF_2 intersects BF_1 at C and AF_1 intersects BF_2 at D. The tangents to $\mathcal{E}$ at A, B intersect at P. Suppose C, D, F_1, F_2 lie on a circle. If $PF_1 = 5$, $F_1F_2 = 7$ and $PF_2 = 8$, then CD can be expressed as $\frac{a}{b}$ for relatively prime positive integers a, b. Compute $a + b$.

Sol :(by **David Altizio**) Let S and T be the reflections of F_1 across AP and BP, respectively. Recall that S, A, and F_2 are collinear, as are T, B, and F_2. Furthermore, $PS = PF_1 = PT$ and $F_2T = F_2S$. It follows that triangles PF_2S and PF_2T are congruent, implying F_2P bisects $\angle BF_2A$. Analogously, F_1P bisects $\angle AF_1B$. Combined with the fact that F_1CF_2D is cyclic, we deduce that $\angle PF_1C + \angle PF_2C = 90°$.

There are several ramifications to the above claim. First, remark that

$$\angle F_1CF_2 = \angle F_1PF_2 + \angle PF_1C + \angle PF_2C = 150°.$$

Second, observe by a similar angle chase that $\angle PSF_2 + \angle PF_2S = 90°$, so $SP \perp PF_2$. From $PS = 5$ and $PF_2 = 8$ we deduce $\sin\angle SF_2P = \frac{5}{\sqrt{89}}$ and $\cos\angle SF_2P = \frac{8}{\sqrt{89}}$, so

$$\sin\angle CF_2D = \sin(2\angle PF_2C) = 2 \cdot \frac{5}{\sqrt{89}} \cdot \frac{8}{\sqrt{89}} = \frac{80}{89}.$$

Finally, Extended Law of Sines on cyclic quadrilateral F_1CF_2 yields

$$14 = \frac{F_1F_2}{\sin\angle F_1CF_2} = \frac{CD}{\sin\angle CF_2D} = \frac{89CD}{80},$$

whence $CD = \boxed{\frac{1120}{89}}$.

> **Problem 13**
>
> Let a, b, c, d be real numbers such that $abcd = -1$ and the following equations hold :
>
> $$|d(a-b)(b-c)(c-a)| = 1$$
>
> $$|a(b-c)(c-d)(d-b)| = 2$$
>
> $$|b(c-a)(a-d)(d-c)| = 6$$
>
> Find the sum of all possible values of $|c(d-a)(a-b)(b-d)|$.

Sol : Plot points $A(a, \frac{1}{a})$, $B(b, \frac{1}{b})$, $C(c, \frac{1}{c})$ and $D(d, \frac{1}{d})$ on the coordinate plane. $abcd = -1$, implies that the points form a orthoentric system, in particular one of the points lies in the interior of the triangle determined by the rest. Now note that that the first expression is twice the area of $\triangle ABC$, the next is twice the area of $\triangle BCD$, and the next of $\triangle CDA$. Hence twice the area of $\triangle ABD$ must either be the sum of all the three values (=9), if C lies in the interior of $\triangle ABD$, or it should be the difference of the largest minus the sum of other two (=3) (if B lies in $\triangle CAD$.) Thus the sum of possible values is $\boxed{12}$.

Problem 14

A positive integer n is said to be a *quadratic residue modulo 97*, if there exists an integer m, such that $m^2 - n$ is divisible by 97 and $97 \nmid n$. Alice randomly picks a triple (a, b, c) of *quadratic residues modulo 97*, where $1 \leq a, b, c \leq 97$ and her choice of triple is equally likely among all possible choices. If the probability that $a + b + c$ is **NOT** a *quadratic residue modulo 97* can be expressed as $\frac{x}{y}$, for relatively prime positive integers x, y, then compute the value of $x + y$.

Sol : One can prove that a QR can be written as a sum of two QRs in 23 ways and in the sum of two NQRs in 24 ways. Now we complementary count. Instead of counting $a + b + c = d$, we count $a + b = c + d$ (as x is a QR iff $-x$ is a QR, since 97 is 1 mod 4. The common value is equal to a QR for 48×23^2 choices, equal to a NQR for 48×24^2 choices and equal to zero for 48^2 choices. Complimentary counting gives the probability to be $\frac{1151}{2304}$, giving an answer of $\boxed{3455}$

Problem 15

Let N denote the number of upright paths from $(0, 0)$ to $(10, 10)$ which intersect the line $x = y$ at exactly 5 points other than the start and end points. If N can be expressed as $2^a \times b$, where b is an odd positive integer, compute $a + b$.

Note : A upright path is one where we only take steps towards right or upwards. For example the following is an upright path from $(0, 0)$ to $(2, 2)$:

$$(0, 0) \to (1, 0) \to (2, 0) \to (2, 1) \to (2, 2)$$

whereas the following is not (as it has a leftward step):

$$(0, 0) \to (1, 0) \to (2, 0) \to (3, 0) \to (3, 1) \to (3, 2) \to (2, 2)$$

Sol : We will solve this using generating functions. Consider the line segment with endpoints at $(0, 0)$ and $(10, 10)$. We are basically dividing this segment into 6 parts. For each sub segment of length $k\sqrt{2}$, we can from one endpoint of it to the other using $2 \times C_{k-1}$ ways. (C_n denotes n-th Catalan Number). So the problem boils down to finding the coefficient of x^{10} in

$$2^6(x + x^2 + 2x^3 + 5x^4 + 14x^5 + \cdots)^6$$

Actually we just need to find the coefficient of x^4 in $(1 + x + 2x^2 + 5x^3 + 14x^4)^6$. After some casework, it comes out to be 429. This gives a final answer of $6 + 429 = \boxed{435}$

3.2 Subjective Problems

Problem 1 (*Kazi Aryan Amin*)
Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that the following equation holds for all $x, y \in \mathbb{R}$:

$$f(yf(x)) + f(xy) = 2yf(x)$$

Proof. The only solutions are $f(x) = x$ for all $x \in \mathbb{R}$ and $f(x) = 0$ for all $x \in \mathbb{R}$. Its easy to see that these work. We now show that these are the only ones.

Let $P(x, y)$ denote the given assertion. Note that $P(0, 0)$ gives $f(0) = 0$

Assume that there exists some $a \neq 0$ with $f(a) = 0$. Then $P(\frac{x}{a}, a)$, gives $f(x) = 0$ for all $x \in \mathbb{R}$, which is indeed a solution.

Now assume $f(x) = 0 \implies x = 0$. We show that f is injective. Indeed suppose $f(a) = f(b)$ for some $a \neq b$. Then $P(a, x)$ and $P(b, x)$ imply $f(ax) = f(bx)$ for all $x \in \mathbb{R}$. And $P(x, a)$ and $P(x, b)$ gives $a = b$, proving injectivity for $f \not\equiv 0$.

Finally, $P(x, y)$ and $P(y, x)$ together give $f(xf(y)) = f(yf(x))$ for all $x, y \in RR$, implying $xf(y) = yf(x)$, so that $\frac{f(x)}{x}$ is constant for $x \neq 0$. This implies $f(x) = cx$ for some $c \in \mathbb{R}$. Plugging back we observe that only $c = 1$ works. $\square$

Problem 2 (*Kazi Aryan Amin*)

Find all pairs of positive integers (a, b) such that for all positive integers $n > 2023^{2023}$, the number $n^2 + an + b$ has a divisor $d > 1$, such that $n \mid d - 1$.

Proof. The answer is all pairs of the form $(a, 1)$, and (a, b) with $a - b = 1$. (where a, b are positive integers. Indeed, if $b = 1$, then $n^2 + an + 1$ is a 1 mod n factor of itself. For $a - b = 1$, $n^2 + an + b$ always has $n + 1$ as a factor.

We now show that no other pairs work. For any such working pair (a, b), and n large, there exists integers $x_n, y_n \geq 0$, such that $n^2 + an + b = (nx_n + 1)(ny_n + b)$.

Now we have two cases :

- $y_n = 0$ for all large enough n. This implies b divides $n^2 + an + b$ for all large n. Hence b divides $(n+1)^2 + a(n+1) + b - n^2 - an - b = 2n + a + 1$ for all large n. This can hold iff $b = 1$ or $b = 2$ and a is odd. In the later case, we have:

$$nx_n + 1 = \frac{n^2 + an + 2}{2} = \frac{n(n+a)}{2} + 1$$

  for all large n. However for $n = 2k$, with k odd, we see that the LHS is odd, whereas the RHS is even, which leads to a contradiction. Hence $b = 1$.

- $y_n \geq 1$ for infinitely many n. In this case, we may bound:

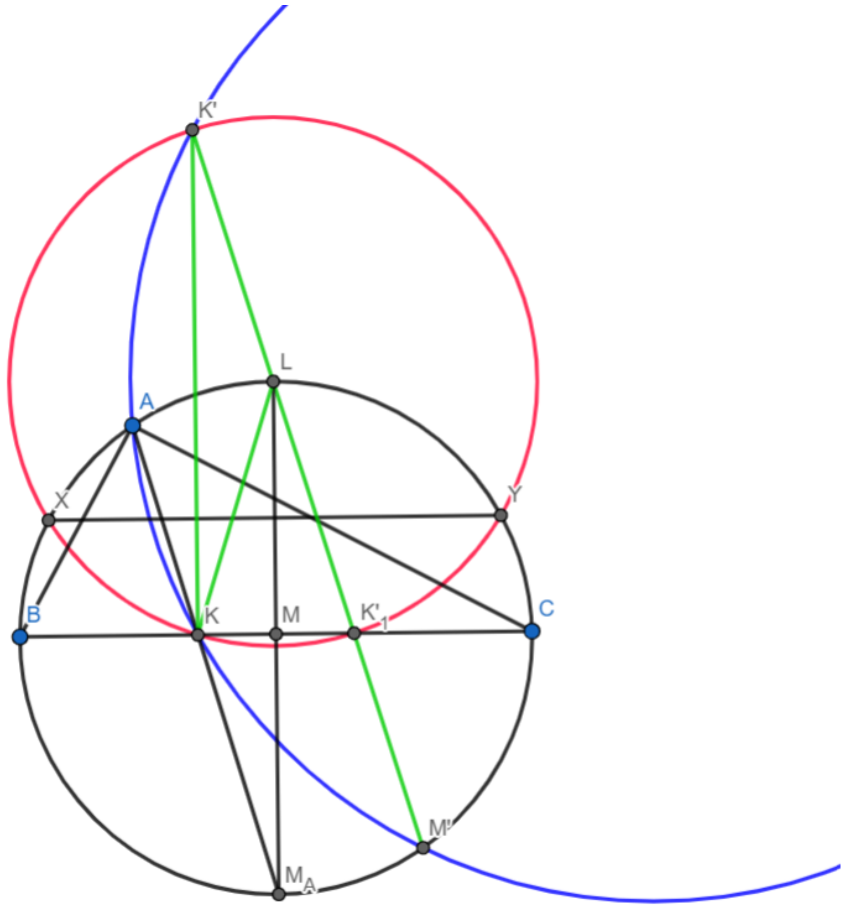
$$(n^2 + an + b) = (nx_n + 1)(ny_n + b) > x_n y_n \cdot n^2$$

  However for sufficiently large n, $n^2 + an + b < 2n^2$, implying $x_n y_n < 2$ for sufficiently large n. Thus $x_n = y_n = 1$, for all large n. This implies $n + 1$ divides $n^2 + an + b$ for infinitely many integers n, thus it must do so as a polynomial as well. This leads to $a - b = 1$.

Having exhausted all cases we are done. $\square$

Problem 3 (*Kazi Aryan Amin*)

Let ω denote the circumcircle of triangle ΔABC. Suppose the internal and external bisectors of $\angle BAC$, intersect $\overline{BC}$ and ω again at K, L respectively. Points X, Y lie on ω such that $LK = LX = LY$. Prove that $\overline{XY}$, and the line through K perpendicular to $\overline{BC}$ meet on the A-median.



Proof.

Let M denote the midpoint of BC and suppose AM hits $\odot(ABC)$ again at M'. Suppose the line through K perpendicular to BC hits $M'L$ at K'. We have the following claims:

Claim. *$M'L$ passes through the reflection K'_1 of K in M*

Define $f(Z) = \pm\frac{BZ}{CZ}$ to be positive if Z and A are on the same side of BC, and negative otherwise. Also define it to be negative on segment BC and positive on other points on line BC. Then by standard applications of the ratio lemma, we have : $f(M')f(A) = f(M) = 1$ and hence $f(LM' \cap BC) = f(L)f(M') = \frac{1}{f(A)} = \frac{1}{f(K)}$ and thus the claim follows.

Claim. *$LK = LK'$*

Angle chase:

$$\angle K'KL = \angle KLM = \angle MLM' = \angle LK'K \implies LK = LK'$$

Claim. *K', A, K, M' lie on a circle*

Note that A, K, M, L are concyclic. Thus we have :

$$\angle KAM' = \angle KLM = \angle MLM' = \angle KK'M'$$

Conclude by radical axis on $\odot(ABC)$, $\odot(KXY)$ and $\odot(K'AKM')$.

$\square$

Problem 4 (*Kazi Aryan Amin* and *Pranjal Srivastava*)
2002 people stand in a line. Each person either always tells the truth, or always lies. Starting from the back, Alice asks 1995 people :

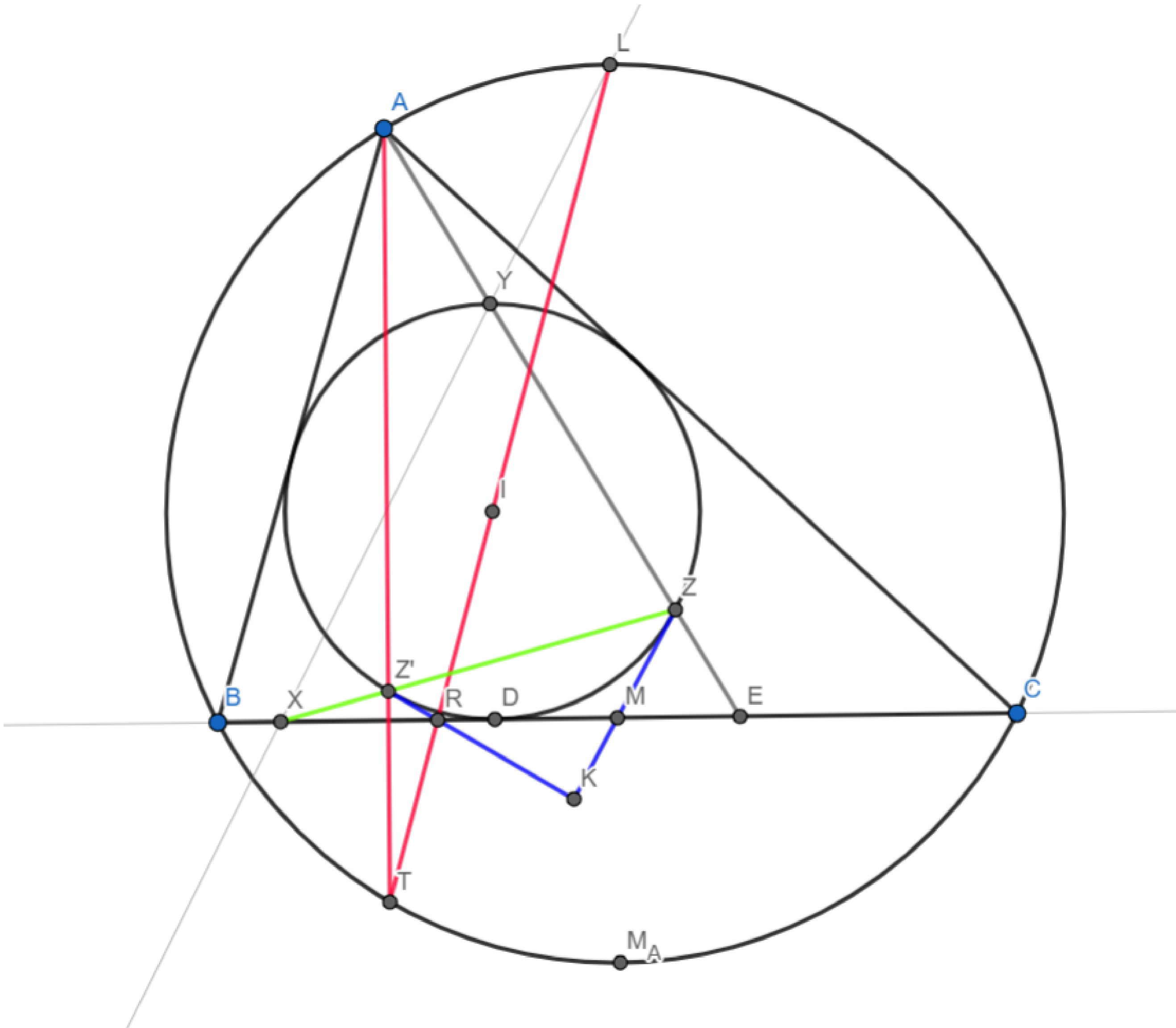
How many liars are are standing in front of you?

and records their answers. Prove Alice can pick a subset of non-negative integers $\mathcal{S}$ such that the sum of elements of $\mathcal{S}$ is atmost 2023 and she can guarantee that number of truthful people is in $\mathcal{S}$.

Proof. Number the people 1 through 2002 from left to right. We start with the set $\mathcal{S} = \{0, 1, 2, \cdots 7\}$, which denotes the number of possible truthful people amongst the first 7 people. Then we start asking from the eighth person onwards, and inductively build our set $\mathcal{S}$. At every stage, consider the answer given by the i-th individual, say x. If $x \notin \mathcal{S}$, then person i must be a liar, as $\mathcal{S}$ must contain the number of truthful people in $[1, i-1]$. If $x \in \mathcal{S}$, then we update x to $x+1$. This works because if i was truthful, the number of truthful people in $[1, i+1]$ should be $x+1$ (in particular, not $=x$). Similarly if i is a liar, then number of truthful people in $[1, i]$ is not equal to x, so we may delete x safely in this case. This covers all cases. Note that the sum of elements of S increases by 1 at each stage, and hence the final sum of elenents of $\mathcal{S}$ is atmost $0 + 1 + 2 + \cdots + 7 + 1995 = 2023$. $\square$

Problem 5 (*Kazi Aryan Amin*)

The A-excircle of ΔABC is tangent to $\overline{BC}$ at D. The line $\overline{AD}$ intersects the incircle of ΔABC at points Y, Z such that $AZ > AY$. The line through Z parallel to the external angle bisector of $\angle BAC$ meets $\overline{BC}$ at X. Prove that $\overline{XY}$ passes through the midpoint of arc $\widehat{BAC}$ in the circumcircle of ΔABC.



Proof.

Rename the A-extouch point to be E and let the incircle be tangent to BC at D. Suppose the reflection of AE over AI hits the incircle again at Z' and the circumcircle of $\triangle ABC$ at T. Let L denote the midpoint of $\widehat{BAC}$ in $\odot(ABC)$. It is well known that T is the tangency point of the A-mixtilinear incircle with the circumcircle of $\triangle ABC$. We now proceed with the following claims:

Claim. *Z' lies on XZ*

Follows from the fact that, $\{Z, Z'\}$ and $\{AE, AT\}$ are reflections about AI.

Claim. *Suppose LI hits BC again at R. Then RZ' is tangent to the incircle*

Note that TD and TA are isogonal in $\angle BTC$. Thus Z' and D are reflections of each other in TL. Since RD is tangent to the incircle, the claim follows.

Now we're ready to finish. Note that it is well known that MZ is tangent to the incircle. Suppose $RZ' \cap MZ = K$, and hence we have $(XD; RM) = -1$, as the incircle is the K-excircle in $\triangle KZZ'$. Again note we have :

$$-1 = (YD; I\infty_{DY}) \overset{L}{=} (LY \cap BC, D; RM)$$

These together imply $LY \cap BC = X$, as desired

$\square$